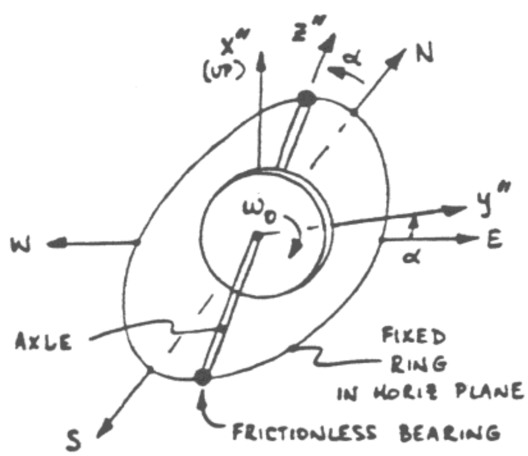
University of California, Berkeley
Physics 105 Fall 2000 Section 2 (*Strovink*)

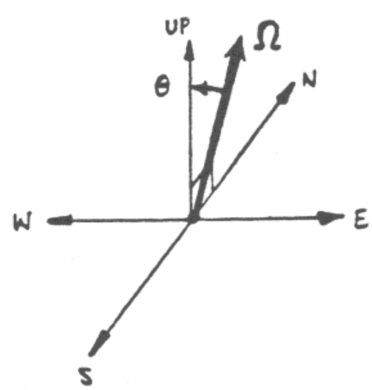
SOLUTION TO PROBLEM SET 10
Solutions by T. Bunn

Reading:
105 Notes 12.1-12.4
Hand & Finch 9.1-9.6

1. and **2.** (double credit problem)
The *Foucault gyrocompass* is a gyroscope that eventually, taking advantage of frictional damping, points to true (not magnetic) north. Thus it is an essential guidance system component.

The gyrocompass may be modeled as a thin disk spinning with angular frequency ω_0 about its symmetry axis z''. This axis can move freely in the horizontal (North-South-East-West) plane only. As exhibited in the following diagrams, the z'' axis makes an angle $\alpha(t)$ with North. The gyrocompass is located at colatitude θ on an earth spinning with angular frequency Ω.





Assuming that $\omega_0 \gg \Omega$ and $\omega_0 \gg \dot{\alpha}$, prove that $\alpha(t)$ oscillates about $\alpha = 0$ provided that $\alpha \ll 1$. Find the angular frequency of oscillation. Note that friction in the bearings will eventually damp out this oscillation, enabling the gyrocompass to point to true north, as defined by the earth's axis of rotation.

You may find the following hints useful:

- Work the problem in the body ($''$) system. This system is obviously not the same as the fixed ($'$) system. It is also not the same as the unprimed system, which is the North-South-East-West system attached to the earth. Using Euler's equations would require knowing the torque from the bearings, evaluated in the body system. Since this torque is not known *a priori*, Euler's equations are not useful here.
- Write $\omega_{x''}$, $\omega_{y''}$, and $\omega_{z''}$ in terms of Ω, α, $\dot{\alpha}$, and θ.
- To get the relationship between the torque $\mathbf{N}'$ applied by the bearings and the angular momentum $\mathbf{L}''$, first write $\mathbf{N}' = d\mathbf{L}'/dt$ (taking advantage of the fact that the ($'$) system is inertial.) Then transform $\mathbf{L}'$ to the $''$ system.
- When evaluating $\mathbf{L}$, remember to neglect terms that are smaller by a factor Ω/ω_0 than the leading terms.

Solution:
One thing that is certainly going to prove useful is the angular velocity vector of the gyroscope. It is the sum of three parts: the rotation $\vec{\omega}_0$ about the gyroscope axis; the rate of change of the gyroscope axis direction, which is equal to $\dot{\alpha}$, and which points in the $\hat{x}''$ direction; and the rotation of the earth, $\vec{\Omega}$. In the $''$ coordinate

system, these all add up to

$$\vec{\omega} = \hat{x}''(\dot{\alpha} + \Omega\cos\theta) + \hat{y}''\Omega\sin\theta\sin\alpha \\ + \hat{z}''(\omega_0 + \Omega\sin\theta\cos\alpha) \ .$$

From this we can find the angular momentum vector in the $''$ system, since the inertia tensor I is diagonal in this system: $\vec{L} = I_1\omega_{x''}\hat{x}'' + I_1\omega_{y''}\hat{y}'' + I_3\omega_{z''}\hat{z}''$.

We want to apply the rule $\vec{N} = \dot{\vec{L}}$, but we can't, because the $''$ system is not inertial. We can, however, say this:

$$\vec{N} = \left(\frac{d\vec{L}}{dt}\right)_{\text{inertial}} = \left(\frac{d\vec{L}}{dt}\right)_{\text{rotating}} + \vec{\omega}^* \times \vec{L}$$

where $\vec{\omega}^*$ is the angular velocity vector of the rotating coordinate system with respect to the inertial coordinate system. Let's consider the $''$ system as our rotating system. Then $\vec{\omega}^*$ is the sum of two components: the rotation of the earth $\vec{\Omega}$, and the rotation of the gyroscope axis, $\dot{\alpha}\hat{x}''$. The components of $\vec{\omega}^*$ in the $''$ system are

$$\vec{\omega}^* = \hat{x}''(\dot{\alpha} + \Omega\cos\theta) + \hat{y}''\Omega\sin\theta\sin\alpha \\ + \hat{z}''\Omega\sin\theta\cos\alpha \ .$$

Now, we're interested in the motion of the gyroscope about the x'' axis, so let's write down the x'' component of our torque equation. There is no torque in this direction (because the bearing is frictionless), so

$$N_{x''} = 0 = \left(\frac{d\vec{L}''}{dt}\right)_{x''} + \left(\vec{\omega}^* \times \vec{L}\right)_{x''}$$

(Note that $(d\vec{L}''/dt)_{x''}$ means the x'' component of the time derivative of $\vec{L}$ as seen in the $''$ system.) We know the components of all of these vectors in the $''$ system, so we can write this expression explicitly:

$$0 = I_1\ddot{\alpha} + I_3\Omega\sin\theta\sin\alpha(\omega_0 + \Omega\sin\theta\cos\alpha) \\ - I_1\Omega^2\sin^2\theta\sin\alpha\cos\alpha \ .$$

To simplify this equation, note that Ω is much smaller than any other frequency in the problem. So let's drop all terms higher than first order in Ω.

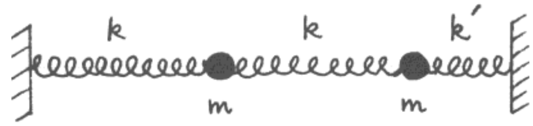
$$I_1\ddot{\alpha} + I_3\omega_0\Omega\sin\theta\sin\alpha = 0$$

Approximate $\sin\alpha \approx \alpha$, and you get the harmonic oscillator equation. α thus oscillates about 0 (true north) with frequency γ, given by

$$\gamma^2 = \frac{I_3}{I_1}\omega_0\,\Omega\sin\theta.$$

3.

Consider a coupled oscillator with two equal masses m, each connected to fixed supports by springs with unequal spring constants k and k'. The two masses are connected to each other by a spring with spring constant k.



Find its two natural angular frequencies.

Solution:

Let x_1 and x_2 be the displacements of the two masses from their equilibrium positions. Then the forces on the two masses are $F_1 = -kx_1 - k(x_1 - x_2)$, and $F_2 = -k'x_2 - k(x_2 - x_1)$. So the equations of motion are

$$m\ddot{x}_1 + 2kx_1 - kx_2 = 0$$
$$m\ddot{x}_2 - kx_1 + (k + k')x_2 = 0$$

Guess that the solutions are periodic: $x_j = A_j e^{i\omega t}$. Then we get a pair of linear equations for A_1 and A_2:

$$\left(2k - m\omega^2\right)A_1 - kA_2 = 0$$
$$-kA_1 + \left(k' + k - m\omega^2\right)A_2 = 0$$

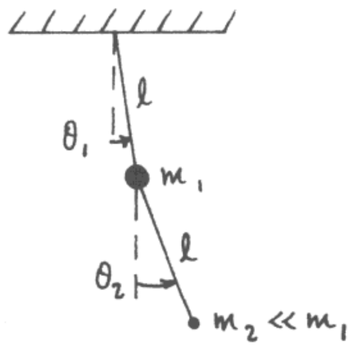
These equations only have nontrivial solutions if the determinant of the coefficients is zero:

$$0 = \begin{vmatrix} 2k - m\omega^2 & -k \\ -k & k + k' - m\omega^2 \end{vmatrix} \\ = m^2\omega^4 - m(3k + k')\omega^2 + k^2 + 2kk'$$

The solutions to this quadratic equation for ω^2 are the two natural frequencies:

$$\omega^2 = \frac{3k + k' \pm \sqrt{5k^2 - 2kk' + k'^2}}{2m}$$

4.
Consider a double pendulum as exhibited in the following diagram. The two pendula are of equal lengths ℓ, but the lower mass $m_2 \ll m_1$. Choose θ_1 and θ_2, the angles between each string and the vertical, as generalized coordinates.



(a)
Find the natural angular frequencies of oscillation.
Solution:
Let's write a Lagrangian. The kinetic energy of mass 1 is easy: $T_1 = \frac{1}{2} m_1 l^2 \dot{\theta}_1^2$. The potential is easy too: $V = -m_1 g l \cos\theta_1 - m_2 g l(\cos\theta_1 + \cos\theta_2)$. Now what about T_2? Well, the position of mass 2 has x, y coordinates $\vec{r}_2 = (l\sin\theta_1 + l\sin\theta_2, -l\cos\theta_1 - l\cos\theta_2)$. Take the time derivative and square to get v_2^2. Then you find that

$$T_2 = \tfrac{1}{2} m_2 l^2 \left(\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1\dot{\theta}_2 \cos(\theta_1 - \theta_2)\right)$$

Putting it all together, we get

$$\begin{aligned}\mathcal{L} = &\tfrac{1}{2} l^2 \big((m_1 + m_2)\dot{\theta}_1^2 + m_2\dot{\theta}_2^2 \\ &+ 2m_2\dot{\theta}_1\dot{\theta}_2 \cos(\theta_1 - \theta_2)\big) \\ &+ (m_1 + m_2) g l \cos\theta_1 + m_2 g l \cos\theta_2 \ .\end{aligned}$$

The two Euler-Lagrange equations are

$$\begin{aligned}0 = &(m_1 + m_2) l^2 \ddot{\theta}_1 + m_2 l^2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) \\ &+ m_2 l^2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) + (m_1 + m_2) g l \sin\theta_1 \\ 0 = &m_2 l^2 \ddot{\theta}_2 + m_2 l^2 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) \\ &- m_2 l^2 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) + m_2 g l \sin\theta_2 \ .\end{aligned}$$

We're clearly not going to get anywhere without making some approximations: Start with the small-angle approximation: Set $\sin\theta = \theta$, $\cos\theta = 1$, and drop all terms with more than one power of θ:

$$\begin{aligned}(m_1 + m_2) l^2 \ddot{\theta}_1 + m_2 l^2 \ddot{\theta}_2 + (m_1 + m_2) g l \theta_1 &= 0 \\ m_2 l^2 (\ddot{\theta}_1 + \ddot{\theta}_2) + m_2 g l \theta_2 &= 0\end{aligned}$$

That's better. Now we solve these differential equations in the usual way: by guessing the answer. Assume solutions of the form

$$\begin{aligned}\theta_1 &= A_1 e^{i\omega t} \\ \theta_2 &= A_2 e^{i\omega t}\end{aligned}$$

Substitute these expressions for θ_1 and θ_2, and you get

$$\begin{aligned}(m_1 + m_2) l (g - l\omega^2) A_1 - m_2 l^2 \omega^2 A_2 &= 0 \\ -m_2 l^2 \omega^2 A_1 + m_2 l (g - l\omega^2) A_2 &= 0\end{aligned} \qquad (1)$$

There's no nontrivial solution unless the determinant of the coefficients is zero:

$$\begin{vmatrix} (m_1 + m_2) l (g - l\omega^2) & -m_2 l^2 \omega^2 \\ -m_2 l^2 \omega^2 & m_2 l (g - l\omega^2) \end{vmatrix} = 0$$

Some notation: Define $\omega_0^2 = g/l$, and $\epsilon = m_2/m_1$. Then computing the determinant and canceling some terms, we get

$$\epsilon(1 + \epsilon)\left(\omega_0^2 - \omega^2\right)^2 - \epsilon^2\omega^4 = 0$$

This equation has two solutions for ω^2, which we'll call ω_+ and ω_-.

$$\omega_\pm^2 = \omega_0^2 \left(1 + \epsilon \pm \sqrt{\epsilon(1 + \epsilon)}\right)$$

These are the "natural frequencies" of this system.
(b)
Calculate the interval $\mathcal{T}/2$ between times for which one or the other bob has minimum amplitude of oscillation. [*Hint:* This is $\pi/\Delta\omega$, where

$\Delta\omega$ is the difference between the two natural angular frequencies.]

Solution:

Rather than accept the hint, which greatly simplifies this part of the problem, why don't we take this opportunity to work out the motion completely. Then the interval $\mathcal{T}/2$ will fall out. First we figure out the amplitudes A_1 and A_2 that go with $\omega_\pm$: From equation (1) above, we get

$$\frac{A_1^\pm}{A_2^\pm} = \frac{\epsilon(\omega_0^2 - \omega_\pm^2)}{\epsilon\omega_\pm^2} = \mp\sqrt{\frac{\epsilon}{1+\epsilon}}$$

(This comes from the second equation in (1), although the first one would have worked just as well. We've skipped some steps in simplifying it.) So far we haven't made the approximation $m_2 \ll m_1$ (*i.e.*, $\epsilon \ll 1$). Let's use it now to say $A_1^\pm/A_2^\pm \approx \mp\sqrt{\epsilon}$. Only the ratio of A_1 to A_2 is determined by the equations of motion, so we can pick the overall magnitude any way we want. Let's say the following:

$$A_1^\pm = \mp\sqrt{\epsilon} \qquad\qquad A_2^\pm = 1$$

The most general solution to the equations of motion will be a linear combination of the + and − solutions:

$$\begin{pmatrix}\theta_1(t)\\ \theta_2(t)\end{pmatrix} = \mathrm{Re}\left(c_+\begin{pmatrix}A_1^+\\ A_2^+\end{pmatrix}e^{i\omega_+ t}\right) + \mathrm{Re}\left(c_-\begin{pmatrix}A_1^-\\ A_2^-\end{pmatrix}e^{i\omega_- t}\right) .$$

Let's say that at $t = 0$, $\dot\theta_1 = \dot\theta_2 = 0$, and $\theta_1 = \theta_2 = \theta_0$. Since the initial velocities are zero, we can take c_+ and c_- to be real, and replace the complex exponentials by cosines. Then our initial conditions say that

$$\begin{pmatrix}(c_- - c_+)\sqrt{\epsilon}\\ c_- + c_+\end{pmatrix} = \begin{pmatrix}\theta_0\\ \theta_0\end{pmatrix}$$

so $c_+ = \frac{1}{2}\theta_0(1-\epsilon^{-1/2})$ and $c_- = \frac{1}{2}\theta_0(1+\epsilon^{-1/2})$. Putting it all together, we get

$$\begin{aligned}\theta_1(t) &= \tfrac{1}{2}\theta_0\Big((1-\epsilon^{1/2})\cos\omega_+ t\\ &\qquad + (1+\epsilon^{1/2})\cos\omega_- t\Big)\\ \theta_2(t) &= \tfrac{1}{2}\theta_0\Big((1-\epsilon^{-1/2})\cos\omega_+ t\\ &\qquad + (1+\epsilon^{-1/2})\cos\omega_- t\Big) .\end{aligned}$$

Now let's return to our original goal of finding the time interval $\mathcal{T}/2$ between maximum and minimum amplitudes of oscillation for one bob. Let's concentrate on θ_1. At $t = 0$, the two terms in the expression for θ_1 are in phase with each other. After a certain time, since $\omega_+ \neq \omega_-$, the two terms will be 180° out of phase, and the amplitude will be minimized. This happens after a time $\mathcal{T}/2 = \pi/(\omega_+ - \omega_-)$. Making our usual small-ϵ argument, the frequency difference is

$$\begin{aligned}\frac{\omega_+ - \omega_-}{\omega_0} &= \sqrt{1+\epsilon+\sqrt{\epsilon(1+\epsilon)}}\\ &\quad - \sqrt{1+\epsilon-\sqrt{\epsilon(1+\epsilon)}}\\ &\approx 1 + \tfrac{1}{2}\epsilon + \tfrac{1}{2}\sqrt{\epsilon(1+\epsilon)}\\ &\quad - 1 - \tfrac{1}{2}\epsilon + \tfrac{1}{2}\sqrt{\epsilon(1+\epsilon)}\\ &= \sqrt{\epsilon(1+\epsilon)}\\ &\approx \sqrt{\epsilon} .\end{aligned}$$

So the time between maximum and minimum oscillation of mass 1 is $\mathcal{T}/2 \approx \pi/\omega_0\sqrt{\epsilon}$.

5.

Consider a linear triatomic molecule, as in the diagram below. A mass M is connected to two masses m, one on either side, by springs of equal spring constant k.

(a)

Find the three natural frequencies of the linear triatomic molecule.

Solution:

If x_1, x_2, x_3 are the displacements of the three atoms from their equilibrium positions, then the equations of motion are

$$\begin{aligned}m\ddot{x}_1 + k(x_1 - x_2) &= 0\\ M\ddot{x}_2 + k(2x_2 - x_1 - x_3) &= 0\\ m\ddot{x}_3 + k(x_3 - x_2) &= 0\end{aligned}$$

As usual, guess that the solutions have time dependence $e^{i\omega t}$. Then there are only solutions if the secular determinant is zero:

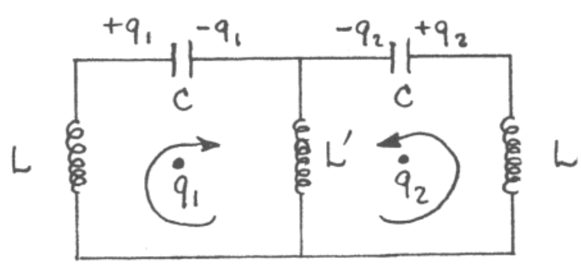
$$0 = \begin{vmatrix} k - m\omega^2 & -k & 0 \\ -k & 2k - M\omega^2 & -k \\ 0 & -k & k - m\omega^2 \end{vmatrix}$$
$$= -\omega^2 \big(Mm^2\omega^4 - 2k(Mm + m^2)\omega^2 + k^2(M + 2m) \big) \ .$$

This cubic equation for ω^2 has three solutions:

$$\omega^2 = 0 \quad \omega^2 = \frac{k}{m} \quad \omega^2 = \frac{k}{Mm}(M + 2m)$$

(**b**)
One of these frequencies should be zero. To what motion does it correspond?
Solution:
The zero-frequency solution corresponds to uniform translation of the molecule, with no stretching of the springs at all.

6.
In a series LC circuit, choose the charge q and its first derivative $\dot{q}$ as independent variables. Equate the "kinetic energy" T to $\frac{1}{2}L\dot{q}^2$ and the "potential energy" U to $\frac{1}{2}q^2/C$. Then Lagrange's equations produce the usual differential equation for the circuit.



In analogy with this approach, find the resonant frequencies of the above LC circuit. Do not rely on loop equations or any other circuit theory. Instead, write the analogous circuit Lagrangian and solve formally using coupled oscillator methods.
Solution:
The Lagrangian for this system is

$$\mathcal{L} = \tfrac{1}{2}L\left(\dot{q}_1^2 + \dot{q}_2^2\right) + \tfrac{1}{2}L'\left(\dot{q}_1 + \dot{q}_2\right)^2 - \frac{1}{2C}\left(q_1^2 + q_2^2\right)$$

which gives equations of motion

$$(L + L')\ddot{q}_1 + L'\ddot{q}_2 + q_1/C = 0$$
$$L'\ddot{q}_1 + (L + L')\ddot{q}_2 + q_2/C = 0$$

The secular determinant is

$$0 = \begin{vmatrix} \frac{1}{C} - (L + L')\omega^2 & -L'\omega^2 \\ -L'\omega^2 & \frac{1}{C} - (L + L')\omega^2 \end{vmatrix}$$
$$= \left((L + L')\omega^2 - \frac{1}{C} \right)^2 - L'^2\omega^4 \ .$$

The solutions are

$$\omega^2 = \frac{1}{LC} \qquad \omega^2 = \frac{1}{(L + 2L')C} \ .$$

7.
Consider a thin homogeneous plate of mass M which lies in the $x_1 - x_2$ plane with its center at the origin. Let the length of the plate be $2A$ (in the x_2 direction) and let the width be $2B$ (in the x_1 direction). The plate is suspended from a fixed support by four springs of equal force constant k located at the four corners of the plate. The plate is free to oscillate, but with the constraint that its center must remain on the x_3 axis. Thus, there are 3 degrees of freedom: (1) vertical motion, with the center of the plate moving along the x_3 axis; (2) a tipping motion lengthwise, with the x_1 axis serving as an axis of rotation (choose an angle θ to describe this motion); and (3) a tipping motion sidewise, with the x_2 axis serving as an axis of rotation (choose an angle ϕ to describe this motion).
(**a**)
Assume only small oscillations and show that the secular equation has a double root and, hence, that the system is degenerate.
Solution:
Let's choose generalized coordinates as follows: Let z be the height of the center of mass, and θ and ϕ be angles of rotation about the x_1 and x_3 axes. Then the kinetic energy is

$$T = \tfrac{1}{2}M\dot{z}^2 + \tfrac{1}{2}I_1\dot{\theta}^2 + \tfrac{1}{2}I_2\dot{\phi}^2$$

The potential energy stored in each spring is just $\frac{1}{2}k$ times the height^2 of the corresponding corner of the slab. The height of the corner in the first quadrant of the x_1-x_2 plane is $z + A\theta - B\phi$ for small θ and ϕ. There are similar expressions with different $+$ and $-$ signs for the other three heights, giving

$$\begin{aligned} V &= \tfrac{1}{2}k\big((A\theta - B\phi + z)^2 + (A\theta + B\phi + z)^2 \\ &\quad + (-A\theta + B\phi + z)^2 + (-A\theta - B\phi + z)^2\big) \\ &= 2k\left(A^2\theta^2 + B^2\phi^2 + z^2\right) \ . \end{aligned}$$

Now set $\mathcal{L} = T - V$ and get the Euler-Lagrange equations

$$\begin{aligned} M\ddot{z} + 4kz &= 0 \\ I_1\ddot{\theta} + 4A^2k\theta &= 0 \\ I_2\ddot{\phi} + 4B^2k\phi &= 0 \end{aligned}$$

The secular determinant is pretty easy:

$$\begin{aligned} 0 &= \begin{vmatrix} 4k - M\omega^2 & 0 & 0 \\ 0 & 4kA^2 - I_1\omega^2 & 0 \\ 0 & 0 & 4kB^2 - I_2\omega^2 \end{vmatrix} \\ &= (4k - M\omega^2)(4kA^2 - I_1\omega^2)(4kB^2 - I_2\omega^2) \ . \end{aligned}$$

The natural frequencies are

$$\omega_1^2 = 4k/M \quad \omega_2^2 = 4kA^2/I_1 \quad \omega_3^2 = 4kB^2/I_2$$

The moments of inertia are $I_1 = \frac{1}{3}MA^2$ and $I_2 = \frac{1}{3}MB^2$, so the last two frequencies are the same: $\omega_2^2 = \omega_3^2 = 12k/M$.

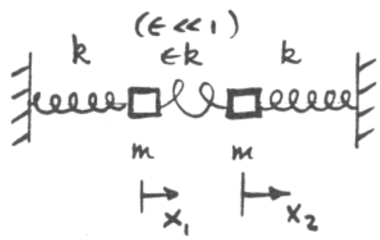
(b)
Discuss the normal modes of the system.
Solution:
The normal modes associated with these three roots of the secular equation are as follows: (1) $\theta = \phi = 0$, $z \propto \cos\omega t$. (Vertical motion; no twisting.) (2) $\phi = z = 0$, $\theta \propto \cos\omega t$. (Rotation about the x_1 axis.) (3) $\theta = z = 0$, $\phi \propto \cos\omega t$. (Rotation about the x_2 axis.) Of course, since modes 2 and 3 are degenerate (*i.e.*, have the same frequency), any linear combination of them could also be chosen as a normal mode.

(c)
Show that the degeneracy can be removed by the addition to the plate of a thin bar of mass m and length $2A$ which is situated (at equilibrium) along the x_2 axis. Find the new eigenfrequencies of the system.
Solution:
If we add a bar of mass m along the x_2 axis, then I_2 is unchanged, while $I_1 = \frac{1}{3}(M+m)A^2$. The three natural frequencies are

$$\begin{aligned} \omega_1^2 &= \frac{4k}{M+m} \\ \omega_2^2 &= \frac{4kA^2}{I_1} = \frac{12k}{M+m} \\ \omega_3^2 &= \frac{4kB^2}{I_2} = \frac{12k}{M} \end{aligned}$$

Since $I_1 \neq I_2$, there is no degeneracy.

8.
Consider a pair of equal masses m connected to walls by equal springs with spring constant k. The two masses are connected to each other by a much weaker spring with spring constant ϵk, where $\epsilon \ll 1$. Choose x_1 and x_2, the displacements from equilibrium of the two masses, as the generalized coordinates.



For this system, write...
(a)
...the spring constant matrix $\mathcal{K}$ and the mass matrix $\mathcal{M}$
Solution:
The Lagrangian for this system can be written as:

$$\begin{aligned} \mathcal{L} &= \left(\frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2\right) \\ &\quad - \left(\frac{1}{2}kx_1^2 + \frac{1}{2}kx_2^2 + \frac{1}{2}\epsilon k(x_1 - x_2)^2\right) \\ &= \frac{1}{2}\left(m\dot{x}_1^2 + m\dot{x}_2^2\right) \\ &\quad - \frac{1}{2}\left(k(1+\epsilon)x_1^2 k(1+\epsilon)x_2^2 - 2\epsilon k x_1 x_2\right) \\ &= \frac{1}{2}\dot{\mathbf{x}} \cdot \mathcal{M}\dot{\mathbf{x}} - \frac{1}{2}\mathbf{x} \cdot \mathcal{K}\mathbf{x} \end{aligned}$$

where

$$\mathcal{M} = m \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$\mathcal{K} = k \begin{pmatrix} 1+\epsilon & -\epsilon \\ -\epsilon & 1+\epsilon \end{pmatrix}$$

(**b**)
...the normal frequencies ω_1 and ω_2
Solution:
Normal frequencies are given by:

$$\det\left(\mathcal{K} - \omega^2 \mathcal{M}\right) = 0$$
$$\begin{vmatrix} k(1+\epsilon) - m\omega^2 & -\epsilon k \\ -\epsilon k & k(1+\epsilon) - m\omega^2 \end{vmatrix} = 0$$

which yields

$$\omega_1^2 = \frac{k}{m}$$
$$\omega_2^2 = \frac{k + 2\epsilon k}{m}$$

(**c**)
...the normal mode vectors $\tilde{a}_1$ and $\tilde{a}_2$ (corresponding to ω_1 and ω_2), each expressed as a linear combination of x_1 and x_2
Solution:
The normal mode vectors $\vec{a}_1$ and $\vec{a}_2$ are determined by the conditions

$$\left(\mathcal{K} - \omega_i^2 \mathcal{M}\right) \vec{a}_i = 0$$
$$\vec{a}_i \cdot \mathcal{M} \vec{a}_i = 1$$

applied using each normal mode frequency in turn. This yields:

$$\vec{a}_1 = \frac{1}{\sqrt{2m}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$\vec{a}_2 = \frac{1}{\sqrt{2m}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

(**d**)
...the 2×2 matrix $\mathcal{A}$ which reduces $\mathcal{M}$ to the unit matrix via the congruence transformation

$$\mathcal{I} = \mathcal{A}^t \mathcal{M} \mathcal{A} \,,$$

where $\mathcal{I}$ is the identity matrix
Solution:
From Eq. 12.13 in the notes, $\mathcal{A}$ is the matrix of normal mode vectors:

$$\mathcal{A} = \frac{1}{\sqrt{2m}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

(**e**)
...the normal coordinates Q_1 and Q_2 (corresponding to ω_1 and ω_2), each expressed as a linear combination of x_1 and x_2.
Solution:
From Eq. 12.15 in the notes:

$$\begin{aligned} \vec{Q} = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} &= \mathcal{A}^t \mathcal{M} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= \frac{1}{\sqrt{2m}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} m \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= \sqrt{\frac{m}{2}} \begin{pmatrix} x_1 + x_2 \\ x_1 - x_2 \end{pmatrix} \end{aligned}$$

So $Q_1 = \sqrt{\frac{m}{2}}(x_1 + x_2)$ and $Q_2 = \sqrt{\frac{m}{2}}(x_1 - x_2)$.